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Darboux transformations and the symmetric fourth Painlevé equation

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Abstract

This paper is concerned with the group symmetries of the fourth Painlevé equation P_{IV}, a second-order nonlinear ordinary differential equation. It is well known that the parameter space of P_{IV} admits the action of the extended affine Weyl group $\tilde{A}_2^{(1)}$. As shown by Noumi and Yamada, the action of $\tilde{A}_2^{(1)}$ as Bäcklund transformations of P_{IV} provides a derivation of its symmetric form SP_4. The dynamical system SP_4 is also equivalent to the isomonodromic deformation of an associated three-by-three matrix linear system (Lax pair). The action of the generators of $\tilde{A}_2^{(1)}$ on this Lax pair is derived using the Darboux transformation for an associated third-order operator.

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1. $\tilde{A}_2^{(1)}$ symmetry and SP_4

Around the beginning of the 20th century Painlevé and his colleagues sought to classify all second-order ordinary differential equations (ODEs) of the form

$$\frac{d^2\eta}{d\zeta^2} = R\left(\zeta, \eta, \frac{d\eta}{d\zeta}\right),$$

having the Painlevé property, which is that their general solutions are absent of movable critical points, see [23] and [4, 11, 15, 16, 20, 29] also. Of the 50 generic types, 6 are not solvable in terms of previously known functions and are now known as the Painlevé equations, which we shall label P_I–P_{VI}, respectively. Our concern is with P_{IV},

$$P_{IV}: \quad \frac{d^2\eta}{d\zeta^2} = \frac{1}{2\eta}\left(\frac{d\eta}{d\zeta}\right)^2 + \frac{3}{2}\eta^3 + 4\zeta\eta^2 + 2\eta(\zeta^2 - \alpha) + \frac{\beta}{\eta}, \tag{1.1}$$

where $\alpha, \beta \in \mathbb{C}$. The Painlevé equations have experienced a renaissance since the discovery of the inverse scattering transform [19]. The revival in interest is due to the fact that P_I–P_{VI}

are all symmetry reductions of completely integrable partial differential equations [1, 2, 7]. For example, the Boussinesq equation

$$U_{tt} + \tfrac{2}{3}(U^2)_{xx} + \tfrac{1}{3}U_{xxxx} = 0 \tag{1.2}$$

has solutions expressible in terms of $\mathrm{P_{IV}}$ via the symmetry reduction

$$U(x,t) = -\frac{1}{4t}\left[\frac{9}{2}\left(\frac{\mathrm{d}\eta}{\mathrm{d}\zeta} + \eta^2 + 2\zeta\eta\right) + 7\zeta^2 - 3(\alpha - 1)\right], \qquad \zeta = \frac{3^{1/2}}{2}xt^{-1/2},$$

see [10]. Furthermore, (1.2) is considered to be a completely integrable partial differential equation as it can be derived from the following Lax pair:

$$\begin{aligned} \mathcal{L}\psi &= \lambda\psi, \qquad & \mathcal{L} &= \partial_x^3 + U(x,t)\partial_x + V(x,t), \\ \psi_t &= \mathcal{M}\psi, \qquad & \mathcal{M} &= \partial_x^2 + W(x,t), \end{aligned} \tag{1.3}$$

see [39]. As well as their physical applications the Painlevé equations have many mathematical properties, of which we are mainly concerned with Lax pairs [1, 12, 19] and Bäcklund transformations (BTs) [4, 8, 13, 14, 20, 21, 26, 27, 35].

From the works of Okamoto [35], it is known that the parameter space of $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ all admit the action of an extended affine Weyl group—specifically the group acts as a group of BTs. In the case of $\mathrm{P_{IV}}$ the group is $\widetilde{A}_2^{(1)}$, see definition 2.1. This idea has been applied in a series of recent articles [31–34] by Noumi and Yamada to rederive the symmetric form of $\mathrm{P_{IV}}$,

$$\frac{\mathrm{d}f_0}{\mathrm{d}x} = f_0(f_1 - f_2) + \alpha_0, \tag{1.4a}$$

$$\mathrm{SP_4}: \quad \frac{\mathrm{d}f_1}{\mathrm{d}x} = f_1(f_2 - f_0) + \alpha_1, \tag{1.4b}$$

$$\frac{\mathrm{d}f_2}{\mathrm{d}x} = f_2(f_0 - f_1) + \alpha_2. \tag{1.4c}$$

From here on we shall use x to denote the independent variable in $\mathrm{SP_4}$, which should not be confused with the spatial variable of the Boussinesq equation. $\mathrm{SP_4}$ is the first member of a hierarchy of symmetric dynamical systems related to $\widetilde{A}_N^{(1)}$, $N \geqslant 2$, also containing the symmetric form of $\mathrm{P_V}$ ($\mathrm{SP_5}$) associated with $\widetilde{A}_3^{(1)}$. Recasting $\mathrm{P_{IV}}$ and $\mathrm{P_V}$ into their symmetric forms allows one to understand their group symmetries naturally. Although $\mathrm{SP_4}$ was known to Bureau [6], and both $\mathrm{SP_4}$ and $\mathrm{SP_5}$ to Adler [3], as alternative representations of these ODEs, they were never exploited to understand these group symmetries. Up to a scaling of the variables, $\mathrm{SP_4}$ is equivalent to $\mathrm{P_{IV}}$ in any of the unknown functions f_j. For example, first scaling the system

$$f_j = (-1/2)^{1/2}\eta_j(\zeta), \qquad x - x_0 = (-2)^{1/2}\zeta, \qquad j = 0, 1, 2,$$

for arbitrary $x_0 \in \mathbb{C}$ and then eliminating η_1 and η_2 gives

$$\eta_0 \text{ satisfies } \mathrm{P_{IV}}: \quad \alpha = \alpha_1 - \alpha_2, \qquad \beta = -2\alpha_0^2. \tag{1.5}$$

Similar results for η_1 and η_2 can be calculated using (1.5) and the permutation symmetry of $\mathrm{SP_4}$, thus working with $\mathrm{SP_4}$ is equivalent to working with three copies of $\mathrm{P_{IV}}$ simultaneously. The symmetries of $\mathrm{P_{IV}}$ are encompassed in the general framework of the extended affine Weyl group $\widetilde{A}_2^{(1)}$, which acts on $\mathrm{SP_4}$ as a group of BTs, see theorem 2.1.

We wish to understand the group symmetries of $\mathrm{SP_4}$ on the level of its associated Lax pair by implementing the Darboux transformation (DT) methodology, see [25] and [3, 9, 22, 37, 38]. In general terms, any coupled system of linear matrix ODEs of the form

$$\mathbf{\Psi}_z(x,z;\mathbf{v}) = \widetilde{\mathbf{M}}(x,z;\mathbf{v})\mathbf{\Psi}(x,z;\mathbf{v}), \tag{1.6a}$$

$$\mathbf{\Psi}_x(x,z;\mathbf{v}) = \mathbf{L}(x,z;\mathbf{v})\mathbf{\Psi}(x,z;\mathbf{v}), \tag{1.6b}$$

where $\mathbf{\Psi}$ is a vector of eigenfunctions, $\mathbf{v}$ is a vector of parameters and $\widetilde{\mathbf{M}}, \mathbf{L}$, are square matrices, is known as a Lax pair. One can associate with each Lax pair a nonlinear differential equation in the following way: imposing the compatibility condition $\mathbf{\Psi}_{xz} = \mathbf{\Psi}_{zx}$ yields the zero curvature equation

$$\frac{\partial\widetilde{\mathbf{M}}}{\partial x} - \frac{\partial\mathbf{L}}{\partial z} + [\widetilde{\mathbf{M}}, \mathbf{L}] = 0. \tag{1.7}$$

For a suitable choice of $\widetilde{\mathbf{M}}$ and $\mathbf{L}$, (1.7) is equivalent to a nonlinear differential equation. In [26], a $sl(2,\mathbb{C})$ Lax pair was used to study $\mathrm{P_{IV}}$. However, as $\widetilde{A}_2^{(1)}$ is the affine Weyl group of $sl(3,\mathbb{C})$ the group symmetries of $\mathrm{P_{IV}}$ are not explicit. For this reason, we approach $\mathrm{P_{IV}}$ by studying the Lax pair of $\mathrm{SP_4}$ instead. To be consistent with [34], we slightly modify the general Lax pair above, so rather than $\widetilde{\mathbf{M}}$ we shall use $\mathbf{M} = z\widetilde{\mathbf{M}}$, giving the following Lax matrices for $\mathrm{SP_4}$:

$$\mathbf{M} = -\begin{pmatrix} v_1 & f_1 & 1 \\ z & v_2 & f_2 \\ zf_0 & z & v_3 \end{pmatrix}, \qquad \mathbf{L} = -\begin{pmatrix} g_1 & 1 & 0 \\ 0 & g_2 & 1 \\ z & 0 & g_3 \end{pmatrix}, \tag{1.8}$$

where $f_j = f_j(x)$, $g_j = g_j(x)$, $\mathbf{v} = (v_1, v_2, v_3)^T$, $v_j \in \mathbb{C}$ and $\mathbf{\Psi} = (\psi_1, \psi_2, \psi_3)^T$. A routine calculation shows that substituting (1.8) into (1.7) gives $\mathrm{SP_4}$ along with the following relations:

$$\alpha_0 = 1 - v_1 + v_3, \qquad \alpha_1 = v_1 - v_2, \qquad \alpha_2 = v_2 - v_3, \tag{1.9a}$$

$$g_1 = \frac{x}{3} - f_2, \qquad g_2 = \frac{x}{3} - f_0, \qquad g_3 = \frac{x}{3} - f_1. \tag{1.9b}$$

We can choose the normalization $\mathrm{tr}(\mathbf{M}) = \mathrm{tr}(\mathbf{L}) = 0$, since this choice of gauge does not affect $\mathrm{SP_4}$.

The remainder of this paper unfolds as follows: in section 2 we give a formal definition of $\widetilde{A}_2^{(1)}$ and how it acts on $\mathrm{SP_4}$, as presented in [31]. A detailed discussion of gauge transformations corresponding to various elements of $\widetilde{A}_2^{(1)}$ is also presented. The first example of such a gauge transformation is derived in section 3, together with the analysis of a specific gauged system. Using the results at the end of section 3 and the tools developed in section 2, several DTs of (1.6) are calculated in section 4. The DTs given in section 4 are combined to derive a new Schlesinger transformation (ST) of $\mathrm{P_{IV}}$, see [28], as well as then being 'decomposed' in section 5 to derive theorem 5.1. The main results and open problems are summarized in section 6.

Note to the reader. In this paper, we shall adopt the opposite convention to that used by Noumi and Yamada in [30], see also [31–34]: we use right actions of the affine Weyl group $\widetilde{A}_2^{(1)}$ throughout where they use left actions instead; consult the presentation of $\widetilde{A}_2^{(1)}$ given in section 2.

2. $\widetilde{A}_2^{(1)}$ and Lax pairs

The symmetric forms presented in [31] were derived by considering the following problem: 'for a general affine root system, find a class of nonlinear differential (or difference) equations on which the affine Weyl group acts as a group of BTs'. The case of $\widetilde{A}_N^{(1)}$, $N \geqslant 2$, gave rise to the hierarchy of differential systems which contain $\mathrm{SP_4}$ and $\mathrm{SP_5}$ as the first two members. The rest of this section is concerned with the formal definition of $\widetilde{A}_2^{(1)}$, together with a thorough discussion of how one would apply gauge transformations associated with the generators of $\widetilde{A}_2^{(1)}$ to a given Lax pair. Note that the discussion pertaining to Lax pairs is completely general and is not restricted to the three-by-three Lax pair for SP_4.

Definition 2.1. *The extended affine Weyl group* $\widetilde{A}_2^{(1)} = \langle s_0, s_1, s_2, \pi\rangle$ *is defined by the fundamental relations*

$$s_i{}^2 = I, \qquad (s_i s_j)^3 = I, \quad j = i \pm 1, \qquad (s_i s_j)^2 = I, \quad j \neq i, i \pm 1, \tag{2.1a}$$

$$\pi^3 = I, \tag{2.1b}$$

$$s_i \pi = \pi s_{i+1}, \qquad i = 0, 1, 2. \tag{2.1c}$$

With this definition in place we now formalize the relation between $\widetilde{A}_2^{(1)}$ and the BTs of SP_4.

Theorem 2.1 ([31])**.** *The BTs of* SP_4 *generated by* $\widetilde{A}_2^{(1)}$ *is defined by the fundamental relations (2.1) and realized as a group of automorphisms of the field of rational functions* $\mathbb{C}(\alpha_j, f_j)$ *as follows:*

$$s_i(\alpha_i) = -\alpha_i, \qquad s_i(\alpha_j) = \alpha_j + \alpha_i, \qquad j = i \pm 1, \tag{2.2a}$$

$$s_i(f_i) = f_i, \qquad s_i(f_j) = f_j \pm \frac{\alpha_i}{f_i}, \qquad j = i \pm 1, \tag{2.2b}$$

$$\pi(\alpha_j) = \alpha_{j+1}, \qquad \pi(f_j) = f_{j+1}, \qquad j = 0, 1, 2. \tag{2.2c}$$

Trivial actions are not presented and group elements act from the right, so we shall interpret πs_{i+1} as apply π first and then s_{i+1}—this convention will be used throughout. Note that this is the opposite convention to the one taken by Noumi and Yamada, who use left actions [30–34]. It is also worth noting that the action of the Weyl group on the space of solutions of SP_4 is dual to the action on the variables, see the discussion in appendix A.4 in [30].

A natural question to ask would be: how does $\widetilde{A}_2^{(1)}$ act on the Lax pair (1.6) of SP_4? Or equivalently, consider the following problem: derive a set of matrix transformations corresponding to each of the generators of $\widetilde{A}_2^{(1)}$ such that when they are applied to $\mathbf{\Psi}(z, f_j; \mathbf{v})$, this results in the action of that generator on each entry of $\mathbf{L}(z, f_j; \mathbf{v})$ and $\mathbf{M}(z, f_j; \mathbf{v})$. (We have denoted the dependence on x implicitly through the variables f_j for the current discussion.) Furthermore, given a set of matrix transformations representing the action of the generators of $\widetilde{A}_2^{(1)}$, we are then faced with the next problem: how does one multiply these matrix transformations together? This may seem of little importance at the moment, but it transpires to be a crucial point for the following reason: it is plausible that the matrix transformations in question may have dependences on elements such as α_i or f_i. If true, then a sense of order of application of elements belonging to $\widetilde{A}_2^{(1)}$ is essential as the entries of these matrix transformations will change too, see theorem 2.1.

For the sake of the immediate argument assume that a_j are particular elements of the extended affine Weyl group $\widetilde{A}_2^{(1)}$ whose fundamental relations are governed by (2.1). Assume also that the action of the individual group elements is defined by (2.2). (Examples of a_j would be π or s_j.) The action of the element a_1, say, on the Lax pair will be interpreted as the outcome of a_1 on every entry of the Lax pair. Define the action of the element a_1 on the Lax pair as

$$a_1(\mathbf{\Psi}(z, f_j; \mathbf{v})) := \mathbf{\Psi}(z, a_1(f_j); a_1(\mathbf{v})), \tag{2.3a}$$

$$a_1(\mathbf{M}(z, f_j; \mathbf{v})) := \mathbf{M}(z, a_1(f_j); a_1(\mathbf{v})), \tag{2.3b}$$

$$a_1(\mathbf{L}(z, f_j; \mathbf{v})) := \mathbf{L}(z, a_1(f_j); a_1(\mathbf{v})). \tag{2.3c}$$

Now assume that we have a transformation matrix $\mathbf{A}_1(z, f_j; \mathbf{v})$ which when applied to $\boldsymbol{\Psi}(z, f_j; \mathbf{v})$, see (2.5), gives the action of the element a_1 defined by (2.3). The transformed Lax pair is

$$z[a_1(\boldsymbol{\Psi}(z, f_j; \mathbf{v}))]_z = a_1(\mathbf{M}(z, f_j; \mathbf{v}))a_1(\boldsymbol{\Psi}(z, f_j; \mathbf{v})), \tag{2.4a}$$

$$[a_1(\boldsymbol{\Psi}(z, f_j; \mathbf{v}))]_x = a_1(\mathbf{L}(z, f_j; \mathbf{v}))a_1(\boldsymbol{\Psi}(z, f_j; \mathbf{v})), \tag{2.4b}$$

which we obtain as follows: letting $\mathbf{A}_1(z, f_j; \mathbf{v})$ act on $\boldsymbol{\Psi}(z, f_j; \mathbf{v})$

$$a_1(\boldsymbol{\Psi}(z, f_j; \mathbf{v})) = \boldsymbol{\Psi}(z, a_1(f_j); a_1(\mathbf{v})) = \mathbf{A}_1(z, f_j; \mathbf{v})\boldsymbol{\Psi}(z, f_j; \mathbf{v}), \tag{2.5}$$

the Lax pair (2.4) is calculated by differentiating (2.5) with respect to both x and z and then substituting (1.6), giving

$$a_1(\mathbf{M}(z, f_j; \mathbf{v})) = z\frac{\partial \mathbf{A}_1}{\partial z}\mathbf{A}_1{}^{-1} + \mathbf{A}_1\mathbf{M}\mathbf{A}_1{}^{-1}, \tag{2.6a}$$

$$a_1(\mathbf{L}(z, f_j; \mathbf{v})) = \frac{\partial \mathbf{A}_1}{\partial x}\mathbf{A}_1{}^{-1} + \mathbf{A}_1\mathbf{L}\mathbf{A}_1{}^{-1}. \tag{2.6b}$$

Equations (2.5) and (2.6) represent what is meant by applying the group element a_1 on $\boldsymbol{\Psi}(z, f_j; \mathbf{v})$, $\mathbf{L}(z, f_j; \mathbf{v})$, $\mathbf{M}(z, f_j; \mathbf{v})$, defined by (2.3).

Now let us consider how one applies the group element a_2 on $\boldsymbol{\Psi}(z, f_j; \mathbf{v})$ followed by another element, say a_1. In other words, we wish to derive a matrix transformation which when applied to $\boldsymbol{\Psi}(z, f_j; \mathbf{v})$ gives the action corresponding to the *product* of the elements a_1 and a_2, that is a_2a_1 in $\widetilde{A}_2^{(1)}$, on each entry of $\mathbf{L}(z, f_j; \mathbf{v})$ and $\mathbf{M}(z, f_j; \mathbf{v})$. (Recall that this is a right action.) It transpires that this matrix is *not* simply given by the product of the separate matrices corresponding to a_1 and a_2 and then acting it on $\boldsymbol{\Psi}(z, f_j; \mathbf{v})$ in the sense of (2.5).

Define the action of the multiplication of the group elements a_1 and a_2, a_2a_1, on the Lax pair as

$$a_2a_1(\boldsymbol{\Psi}(z, f_j; \mathbf{v})) := \boldsymbol{\Psi}(z, a_2a_1(f_j); a_2a_1(\mathbf{v})), \tag{2.7a}$$

$$a_2a_1(\mathbf{M}(z, f_j; \mathbf{v})) := \mathbf{M}(z, a_2a_1(f_j); a_2a_1(\mathbf{v})), \tag{2.7b}$$

$$a_2a_1(\mathbf{L}(z, f_j; \mathbf{v})) := \mathbf{L}(z, a_2a_1(f_j); a_2a_1(\mathbf{v})). \tag{2.7c}$$

The transformed Lax pair is thus

$$z[a_2a_1(\boldsymbol{\Psi}(z, f_j; \mathbf{v}))]_z = a_2a_1(\mathbf{M}(z, f_j; \mathbf{v}))a_2a_1(\boldsymbol{\Psi}(z, f_j; \mathbf{v})),$$
$$[a_2a_1(\boldsymbol{\Psi}(z, f_j; \mathbf{v}))]_x = a_2a_1(\mathbf{L}(z, f_j; \mathbf{v}))a_2a_1(\boldsymbol{\Psi}(z, f_j; \mathbf{v})),$$

which is obtained in the following way. Consider the action of the element a_2a_1 on the matrix differential operator $\partial_x - \mathbf{L}$. From the derivation of (2.6*b*), we have

$$a_2(\partial_x - \mathbf{L}) = \mathbf{A}_2(\partial_x - \mathbf{L})\mathbf{A}_2{}^{-1},$$

and therefore

$$\begin{aligned}\partial_x - (a_2a_1)(\mathbf{L}) &= a_1(\partial_x - a_2(\mathbf{L}))\\ &= a_1(\mathbf{A}_2)(\partial_x - a_1(\mathbf{L}))a_1(\mathbf{A}_2)^{-1}\\ &= a_1(\mathbf{A}_2)\mathbf{A}_1(\partial_x - \mathbf{L})\mathbf{A}_1{}^{-1}a_1(\mathbf{A}_2)^{-1}.\end{aligned}$$

The corresponding derivation for $\mathbf{M}$ is analogous that for $\mathbf{L}$. (We are grateful to the referee for the elegant derivation above.) Hence, we arrive at the formula

$$a_2 a_1(\mathbf{\Psi}(z, f_j; \mathbf{v})) = \mathbf{A}_2(z, a_1(f_j); a_1(\mathbf{v}))\mathbf{A}_1(z, f_j; \mathbf{v})\mathbf{\Psi}(z, f_j; \mathbf{v}). \tag{2.8}$$

Substituting $\mathbf{A}_2(z, a_1(f_j); a_1(\mathbf{v}))\mathbf{A}_1(z, f_j; \mathbf{v})$ as the transformation matrix into (2.6) multiplying $\mathbf{\Psi}(z, f_j; \mathbf{v})$ will give the Lax matrices $\mathbf{M}(z, a_2a_1(f_j); a_2a_1(\mathbf{v}))$ and $\mathbf{L}(z, a_2a_1(f_j); a_2a_1(\mathbf{v}))$. These formulae will be used repeatedly to calculate Lax pairs transformed by elements of $\widetilde{A}_2^{(1)}$ in section 4.

A few comments: the group $\widetilde{A}_2^{(1)}$ acts on the Lax pair via transformations of $\mathbf{\Psi}(z, f_j; \mathbf{v})$ of the form (2.5), commonly known as a gauge transformation. These gauge transformations are special in that when acted on $\mathbf{\Psi}(z, f_j; \mathbf{v})$ via (2.5), the corresponding action of the element of $\widetilde{A}_2^{(1)}$ is realized on the entries of $\mathbf{L}(z, f_j; \mathbf{v})$ and $\mathbf{M}(z, f_j; \mathbf{v})$. We are able to act $\widetilde{A}_2^{(1)}$ on the Lax pair to derive transformed Lax pairs such as (2.3) and (2.7). Multiplying elements of $\widetilde{A}_2^{(1)}$ on the level of their associated transformation matrices has the added intricacy appearing in (2.8). In contrast to a linear representation ρ of a group where $\rho(a_2a_1) = \rho(a_2)\rho(a_1)$, here one has to apply the action of the *second* element a_1 to the *first* transformation matrix $\mathbf{A}_2$ before multiplying them together. If the matrices have entries such as α_j or f_j, then the transformations must also change accordingly.

In the next section, we build up the tools required to derive the gauge transformations described above. An associated gauged system proves to be vital to the ensuing calculations and so it is also studied in detail.

3. π symmetry and a gauged system

The current section is comprised of two parts—in the first, we derive the gauge transformation associated with the generator π. In the second, we focus our attention on the operator $\mathcal{L}_1$. The gauged system in the variables $\mathbf{\Phi} = (\phi_1, \phi_2, \phi_3)^T = (\psi_1, \psi_{1,x}, \psi_{1,xx})^T$ is intimately related to $\mathcal{L}_1$ which is the third-order Lax operator for the Boussinesq equation (1.3). Studying the action of the generators of $\widetilde{A}_2^{(1)}$ on $\mathcal{L}_1$ allows one to then construct DTs of (1.6) via $\mathbf{\Phi}$. The method used in section 4 is based on the idea of altering the factorization of operators [17, 18]—in its simplest form *different* factorizations of an operator correspond to *different* eigenfunctions. As we shall see in the next section, using the BTs of SP_4 given in theorem 2.1, altering the factorization of operators provides a systematic method of generating DTs corresponding to the action of these group elements.

Begin by rewriting (1.6*b*) as the following system of equations:

$$(\partial_x + g_1)\psi_1 = -\psi_2, \tag{3.1a}$$

$$(\partial_x + g_2)\psi_2 = -\psi_3, \tag{3.1b}$$

$$(\partial_x + g_3)\psi_3 = -z\psi_1. \tag{3.1c}$$

One may reduce (3.1) to a single third-order operator acting on any one of ψ_1, ψ_2, or ψ_3. Substituting ψ_3 and ψ_2 using (3.1*b*) and (3.1*a*) into (3.1*c*) gives

$$\mathcal{L}_1\psi_1 = z\psi_1, \qquad \mathcal{L}_1 = -(\partial_x + g_3)(\partial_x + g_2)(\partial_x + g_1). \tag{3.2a}$$

Similar expressions for $\mathcal{L}_2\psi_2 = z\psi_2$ and $\mathcal{L}_3\psi_3 = z\psi_3$ are given by simply permuting the indices of (3.2*a*), explicitly

$$\mathcal{L}_2 = \pi(\mathcal{L}_1), \qquad \mathcal{L}_3 = \pi(\mathcal{L}_2) = \pi^2(\mathcal{L}_1), \tag{3.2b}$$

emphasis is placed on the fact that the factorization $\mathcal{L}_j$ corresponds to the eigenfunction ψ_j. To determine a set of transformations on ψ_j which gives the action of the generator π, using (3.1), we set

$$\pi(\psi_1) = -F(\partial_x + g_1)\psi_1, \tag{3.3a}$$

$$\pi^2(\psi_1) = -F(\partial_x + g_2)\pi(\psi_1), \tag{3.3b}$$

$$\pi^3(\psi_1) = -F(\partial_x + g_3)\pi^2(\psi_1), \tag{3.3c}$$

where $F = F(z)$. Since $\pi^3(\psi_1) = \psi_1$, working our way back through (3.3) we get

$$\pi^3(\psi_1) = -F^3(\partial_x + g_3)(\partial_x + g_2)(\partial_x + g_1)\psi_1 = F^3 z\psi_1.$$

Thus, requiring $\pi^3(\psi_1) = \psi_1$ implies that

$$F(z) = z^{-1/3}.$$

Overall, the system (3.1) transformed by the generator π is

$$[\partial_x + \pi(g_1)]\pi(\psi_1) = -\pi(\psi_2), \tag{3.4a}$$

$$[\partial_x + \pi(g_2)]\pi(\psi_2) = -\pi(\psi_3), \tag{3.4b}$$

$$[\partial_x + \pi(g_3)]\pi(\psi_3) = -z\pi(\psi_1). \tag{3.4c}$$

One can calculate immediately from (3.3*a*) and (3.1*a*) that

$$\pi(\psi_1) = -z^{-1/3}(\partial_x + g_1)\psi_1 = z^{-1/3}\psi_2. \tag{3.5a}$$

Using $\pi(g_j) = g_{j+1}$ and equations (3.4*a*), (3.5*a*), (3.1*b*) gives

$$\pi(\psi_2) = -z^{-1/3}(\partial_x + g_2)\psi_2 = z^{-1/3}\psi_3. \tag{3.5b}$$

Finally, (3.4*b*), (3.5*b*) and (3.1*c*) yield

$$\pi(\psi_3) = -z^{-1/3}(\partial_x + g_3)\psi_3 = z^{2/3}\psi_1. \tag{3.5c}$$

Collecting (3.5*a*)–(3.5*c*) gives the following gauge transformation

$$\pi(\mathbf{\Psi}) = \mathbf{T}_0\mathbf{\Psi}, \qquad \mathbf{T}_0 = \begin{pmatrix} 0 & z^{-1/3} & 0 \\ 0 & 0 & z^{-1/3} \\ z^{2/3} & 0 & 0 \end{pmatrix}. \tag{3.6}$$

One can calculate the gauged Lax pair under the action of the element π by substituting $\mathbf{A}_1 = \mathbf{T}_0$ into (2.6*a*) and (2.6*b*).

Now we turn our attention to the gauged system $\mathbf{\Phi}$—one can calculate the action of the generators s_j on $\mathcal{L}_j$ by expanding any one of (3.2), choosing (3.2*a*) we get

$$\mathcal{L}_1 = -\partial_x^3 - \left(2\frac{\mathrm{d}g_1}{\mathrm{d}x} + \frac{\mathrm{d}g_2}{\mathrm{d}x} + g_3(g_1 + g_2) + g_1g_2\right)\partial_x - \left(\frac{\mathrm{d}^2g_1}{\mathrm{d}x^2} + g_1\frac{\mathrm{d}g_2}{\mathrm{d}x} + (g_2 + g_3)\frac{\mathrm{d}g_1}{\mathrm{d}x} + g_1g_2g_3\right).$$

Similar calculations for the remaining factorizations (3.2*b*) yield

$$\mathcal{L}_j = -\partial_x^3 - U_j\partial_x - V_j, \qquad j = 1, 2, 3, \tag{3.7a}$$

$$U_1(g_1, g_2, g_3) = 2\frac{\mathrm{d}g_1}{\mathrm{d}x} + \frac{\mathrm{d}g_2}{\mathrm{d}x} + g_3(g_1 + g_2) + g_1g_2, \tag{3.7b}$$

Table 1. Generators of $\widetilde{A}_2^{(1)}$ on U_1 and V_1.

s_1	s_2	s_0	π
U_1	U_1	$U_1 - \frac{\alpha_0}{f_0{}^2}\frac{\mathrm{d}f_0}{\mathrm{d}x}$	U_2
V_1	V_1	$V_1 + B(\alpha_j, f_j)$	V_2

$$V_1(g_1, g_2, g_3) = \frac{\mathrm{d}^2 g_1}{\mathrm{d}x^2} + g_1\frac{\mathrm{d}g_2}{\mathrm{d}x} + (g_2 + g_3)\frac{\mathrm{d}g_1}{\mathrm{d}x} + g_1 g_2 g_3. \tag{3.7c}$$

Up to t-dependent factors each $\mathcal{L}_j$ is the scattering operator of the Boussinesq equation, see (1.3). Expressions for U_j and V_j, $j = 2, 3$, can be calculated using U_1, V_1 and the action of the generator π. It transpires that U_1 is invariant under the action of the generator s_1, that is

$$s_1(U_1) = U_1(s_1(g_1), s_1(g_2), s_1(g_3)) = U_1.$$

Similarly one can show that V_1 is also invariant under s_1, $s_1(V_1) = V_1$. The action of the remaining generators of $\widetilde{A}_2^{(1)}$ on U_1 and V_1 are presented in table 1; note that B is a polynomial in α_j, f_j and the derivatives of f_j. The invariance of U_1 and V_1 under the action of the generators s_1 and s_2, together with the nontrivial action of s_0, may be due to the choice of factorization (3.2*a*); however this issue is not pursued any further.

Finally, we calculate the gauged system $\mathbf{\Phi}$ and how the generator π acts on it. Consider (1.6*b*) in the variables $\mathbf{\Phi} = (\phi_1, \phi_2, \phi_3)^T = (\psi_1, \psi_{1,x}, \psi_{1,xx})^T$. The first two equations of the dynamical system $\mathbf{\Phi}_x$ are given by $\partial_x(\phi_1) = \phi_2$, $\partial_x(\phi_2) = \phi_3$, together with the third

$$\partial_x(\phi_3) = \psi_{1,xxx} = (-V_1 - z)\phi_1 - U_1\phi_2,$$

calculated using (3.7), gives the matrix system

$$\mathbf{\Phi}_x = \mathbf{P}\mathbf{\Phi}, \qquad \mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -V_1 - z & -U_1 & 0 \end{pmatrix}. \tag{3.8a}$$

We shall also require the gauge transformation $\mathbf{g}$, relating the system for $\mathbf{\Psi}$ to the system for $\mathbf{\Phi}$. Expressing ϕ_2 and ϕ_3 as linear combinations of ψ_j gives

$$\phi_2 = -\psi_2 - g_1\psi_1, \qquad \phi_3 = \left(g_1^2 - \frac{\mathrm{d}g_1}{\mathrm{d}x}\right)\psi_1 + (g_1 + g_2)\psi_2 + \psi_3.$$

We prefer to work with $\mathbf{G} = \mathbf{g}^{-1}$, given by the matrix

$$\mathbf{\Psi} = \mathbf{G}\mathbf{\Phi}, \qquad \mathbf{G} = \begin{pmatrix} 1 & 0 & 0 \\ -g_1 & -1 & 0 \\ g_1 g_2 + \frac{\mathrm{d}g_1}{\mathrm{d}x} & g_1 + g_2 & 1 \end{pmatrix}. \tag{3.8b}$$

One may recover (3.8*a*) from the gauge transformation. Differentiating (3.8*b*) and then substituting $\mathbf{L}$ using (1.8), it is straightforward to show that

$$\mathbf{P} = -\mathbf{G}^{-1}\frac{\mathrm{d}\mathbf{G}}{\mathrm{d}x} + \mathbf{G}^{-1}\mathbf{L}\mathbf{G},$$

providing a useful check. From the gauge (3.8*b*), we have

$$\psi_2 = -g_1\phi_1 - \phi_2, \qquad \psi_3 = \left(\frac{\mathrm{d}g_1}{\mathrm{d}x} + g_1 g_2\right)\phi_1 + (g_1 + g_2)\phi_2 + \phi_3,$$

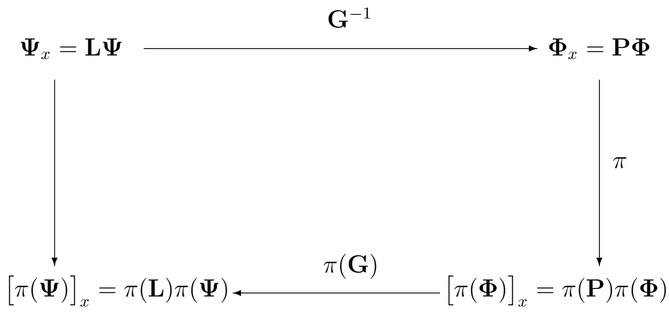

Figure 1. A commutative diagram for DTs.

together with (3.5*a*)–(3.5*c*), concludes this section with $\pi(\mathbf{\Phi})$,

$$\pi(\phi_1) = z^{-1/3}\psi_2 = -z^{-1/3}g_1\phi_1 - z^{-1/3}\phi_2,$$

$$\pi(\phi_2) = z^{-1/3}\psi_{2,x} = -z^{-1/3}\frac{\mathrm{d}g_1}{\mathrm{d}x}\phi_1 - z^{-1/3}g_1\phi_2 - z^{-1/3}\phi_3,$$

$$\begin{aligned}\pi(\phi_3) &= \left(g_2^2 - \frac{\mathrm{d}g_2}{\mathrm{d}x}\right)\pi(\psi_1) + (g_2+g_3)\pi(\psi_2) + \pi(\psi_3),\\ &= z^{-1/3}\left[z - g_1\left(g_2^2 - \frac{\mathrm{d}g_2}{\mathrm{d}x} + \frac{\mathrm{d}g_1}{\mathrm{d}x} + g_1g_2\right)\right]\phi_1\\ &\quad + z^{-1/3}\left[\frac{\mathrm{d}g_2}{\mathrm{d}x} - g_2^2 - g_1^2 - g_1g_2\right]\phi_2 - z^{-1/3}g_1\phi_3.\end{aligned}$$

Collecting the coefficients in the above equations into a matrix $\mathbf{R}$, we can write

$$\pi(\mathbf{\Phi}) = \mathbf{R}\mathbf{\Phi}.$$

The results of this section and section 2 provide us with the tools required to calculate gauge transformations associated with various elements of $\widetilde{A}_2^{(1)}$.

4. Darboux transformations and Lax pairs

The motivation for working with the $\mathbf{\Phi}$ variables becomes clear in this section. We gauge away from the system for $\mathbf{\Psi}$ to the system for $\mathbf{\Phi}$, then apply the action of the generators of $\widetilde{A}_2^{(1)}$ on the $\mathbf{\Phi}$ variables and finally gauge back to the original system. Overall, we expect to obtain the corresponding action on $\mathbf{\Psi}$, our original system. We make repeated use of (2.6) to calculate gauged Lax pairs, as well as (2.8) to derive $\mathbf{X}_j$ below, see section 2 for details. The idea of this method is best illustrated with the use of figure 1, where one may arrive at the transformed Lax pair for $\mathbf{\Psi}$, via $\mathbf{\Phi}$. As a check first we use the method outlined above to rederive $\mathbf{T}_0$. Using figure 1, the results collected in table 1 and the reasoning above, we find that this is in fact the case

$$\pi(\mathbf{\Psi}) = \pi(\mathbf{G})\pi(\mathbf{\Phi}) = \pi(\mathbf{G})\mathbf{R}\mathbf{\Phi} = \pi(\mathbf{G})\mathbf{R}\mathbf{G}^{-1}\mathbf{\Psi} = \mathbf{T}_0\mathbf{\Psi}.$$

The invariance of $\mathbf{\Phi}$ under the action of the generators s_1 and s_2 can be deduced immediately from (3.8*a*), as $s_i(U_1) = U_1$ and $s_i(V_1) = V_1$, $i = 1, 2$, see table 1. Motivated by this fact, in the next calculation before gauging back we act on the $\mathbf{\Phi}$ variables with the generator s_1 after π. Overall we expect to recover the action of the element $s_1\pi$ on the $\mathbf{\Psi}$ variables. Using this reasoning, and following figure 1, we get

$$s_1\pi(\mathbf{\Psi}) = s_1\pi(\mathbf{G})s_1\pi(\mathbf{\Phi}) = s_1\pi(\mathbf{G})\pi(\mathbf{\Phi}) = s_1\pi(\mathbf{G})\mathbf{R}\mathbf{G}^{-1}\mathbf{\Psi}.$$

Overall we arrive at the DT

$$s_1\pi(\boldsymbol{\Psi}) = \mathbf{T}_1\boldsymbol{\Psi}, \tag{4.1a}$$

$$\mathbf{T}_1 = s_1\pi(\mathbf{G})\mathbf{R}\mathbf{G}^{-1} = \begin{pmatrix} 0 & z^{-1/3} & 0 \\ 0 & z^{-1/3}\alpha_2/f_2 & z^{-1/3} \\ z^{2/3} & 0 & 0 \end{pmatrix}, \tag{4.1b}$$

whose elegant simplicity can only be accounted for by the structure of the system. A second application of $s_1\pi$ on (4.1) yields

$$s_1\pi s_1\pi(\boldsymbol{\Psi}) = s_1\pi(\mathbf{T}_1)s_1\pi(\boldsymbol{\Psi}) = \mathbf{T}_2\mathbf{T}_1\boldsymbol{\Psi} = \mathbf{X}_1\boldsymbol{\Psi}, \tag{4.2a}$$

where

$$\mathbf{T}_2 = s_1\pi(\mathbf{T}_1) = \begin{pmatrix} 0 & z^{-1/3} & 0 \\ 0 & z^{-1/3}\frac{\alpha_0+\alpha_2}{f_0+\alpha_2/f_2} & z^{-1/3} \\ z^{2/3} & 0 & 0 \end{pmatrix}, \tag{4.2b}$$

and since $\mathbf{X}_1 = \mathbf{T}_2\mathbf{T}_1$, we get

$$\mathbf{X}_1 = \begin{pmatrix} 0 & z^{-2/3}\alpha_2/f_2 & z^{-2/3} \\ z^{1/3} & z^{-2/3}\frac{\alpha_2(\alpha_0+\alpha_2)}{f_2(f_0+\alpha_2/f_2)} & z^{-2/3}\frac{\alpha_0+\alpha_2}{(f_0+\alpha_2/f_2)} \\ 0 & z^{1/3} & 0 \end{pmatrix}. \tag{4.2c}$$

The entire process is repeated to derive the following DTs $\mathbf{T}_3$–$\mathbf{T}_6$:

$$\mathbf{T}_3 = \begin{pmatrix} 0 & z^{-1/3} & 0 \\ 0 & 0 & z^{-1/3} \\ z^{2/3} & 0 & z^{-1/3}\alpha_0/f_0 \end{pmatrix},$$

$$\mathbf{T}_4 = s_2\pi(\mathbf{T}_3) = \begin{pmatrix} 0 & z^{-1/3} & 0 \\ 0 & 0 & z^{-1/3} \\ z^{2/3} & 0 & z^{-1/3}\frac{\alpha_1+\alpha_0}{f_1+\alpha_0/f_0} \end{pmatrix},$$

$$\mathbf{T}_5 = \begin{pmatrix} z^{-1/3}\alpha_1/f_1 & z^{-1/3} & 0 \\ 0 & 0 & z^{-1/3} \\ z^{2/3} & 0 & 0 \end{pmatrix},$$

$$\mathbf{T}_6 = s_0\pi(\mathbf{T}_5) = \begin{pmatrix} z^{-1/3}\frac{\alpha_2+\alpha_1}{f_2+\alpha_1/f_1} & z^{-1/3} & 0 \\ 0 & 0 & z^{-1/3} \\ z^{2/3} & 0 & 0 \end{pmatrix},$$

which in turn can be multiplied together to give

$$\mathbf{X}_0 = \mathbf{T}_6\mathbf{T}_5 = \begin{pmatrix} z^{-2/3}\frac{\alpha_1(\alpha_2+\alpha_1)}{f_1(f_2+\alpha_1/f_1)} & z^{-2/3}\frac{\alpha_2+\alpha_1}{(f_2+\alpha_1/f_1)} & z^{-2/3} \\ z^{1/3} & 0 & 0 \\ z^{1/3}\alpha_1/f_1 & z^{1/3} & 0 \end{pmatrix},$$

$$\mathbf{X}_2 = \mathbf{T}_4\mathbf{T}_3 = \begin{pmatrix} 0 & 0 & z^{-2/3} \\ z^{1/3} & 0 & z^{-2/3}\alpha_0/f_0 \\ z^{1/3}\frac{\alpha_1+\alpha_0}{(f_1+\alpha_0/f_0)} & z^{1/3} & z^{-2/3}\frac{\alpha_0(\alpha_1+\alpha_0)}{f_0(f_1+\alpha_0/f_0)} \end{pmatrix}.$$

A summary of the DTs listed above is presented in tables 2 and 3.

Table 2. $\mathbf{T}_1, \mathbf{T}_3, \mathbf{T}_5$ and their action on α_j, v_j.

Action	$s_0\pi$	$s_1\pi$	$s_2\pi$
DT	$\mathbf{T}_5$	$\mathbf{T}_1$	$\mathbf{T}_3$
$\begin{pmatrix}\widetilde{v}_1\\ \widetilde{v}_2\\ \widetilde{v}_3\end{pmatrix}$	$\begin{pmatrix}v_1+\frac{1}{3}\\ v_3+\frac{1}{3}\\ v_2-\frac{2}{3}\end{pmatrix}$	$\begin{pmatrix}v_3+\frac{1}{3}\\ v_2+\frac{1}{3}\\ v_1-\frac{2}{3}\end{pmatrix}$	$\begin{pmatrix}v_2+\frac{1}{3}\\ v_1-\frac{2}{3}\\ v_3+\frac{1}{3}\end{pmatrix}$
$\begin{pmatrix}\widetilde{\alpha}_0\\ \widetilde{\alpha}_1\\ \widetilde{\alpha}_2\end{pmatrix}$	$\begin{pmatrix}-\alpha_1\\ \alpha_1+\alpha_2\\ \alpha_1+\alpha_0\end{pmatrix}$	$\begin{pmatrix}\alpha_2+\alpha_1\\ -\alpha_2\\ \alpha_2+\alpha_0\end{pmatrix}$	$\begin{pmatrix}\alpha_0+\alpha_1\\ \alpha_0+\alpha_2\\ -\alpha_0\end{pmatrix}$

Table 3. $\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2$ and their action on α_j, v_j.

Action	$s_0\pi s_0\pi$	$s_1\pi s_1\pi$	$s_2\pi s_2\pi$
DT	$\mathbf{X}_0$	$\mathbf{X}_1$	$\mathbf{X}_2$
$\begin{pmatrix}\widetilde{v}_1\\ \widetilde{v}_2\\ \widetilde{v}_3\end{pmatrix}$	$\begin{pmatrix}v_1+\frac{2}{3}\\ v_2-\frac{1}{3}\\ v_3-\frac{1}{3}\end{pmatrix}$	$\begin{pmatrix}v_1-\frac{1}{3}\\ v_2+\frac{2}{3}\\ v_3-\frac{1}{3}\end{pmatrix}$	$\begin{pmatrix}v_1-\frac{1}{3}\\ v_2-\frac{1}{3}\\ v_3+\frac{2}{3}\end{pmatrix}$
$\begin{pmatrix}\widetilde{\alpha}_0\\ \widetilde{\alpha}_1\\ \widetilde{\alpha}_2\end{pmatrix}$	$\begin{pmatrix}\alpha_0-1\\ \alpha_1+1\\ \alpha_2\end{pmatrix}$	$\begin{pmatrix}\alpha_0\\ \alpha_1-1\\ \alpha_2+1\end{pmatrix}$	$\begin{pmatrix}\alpha_0+1\\ \alpha_1\\ \alpha_2-1\end{pmatrix}$

We close this section with the following remarks: the transformation matrix $\mathbf{T}_0$ is independent of α_j and f_j, so that any element of $\widetilde{A}_2^{(1)}$ acting after π will not alter $\mathbf{T}_0$, specifically $s_j(\mathbf{T}_0) = \mathbf{T}_0$—this turns out to be a crucial point in section 5.

The DTs $\mathbf{X}_j$ are in fact three versions of the same ST for the three-by-three Lax pair (1.6) of the SP_4 system. Roughly speaking a ST is a transformation of the eigenfunctions which produce a shift in the parameters of the corresponding nonlinear ODE. Using (1.5) and table 3, one can verify that $\mathbf{X}_1$ produces the following shift in the parameters of P_{IV}: $\eta_0(\zeta;\alpha,\beta) \to \eta_0(\zeta;\alpha-2,\beta)$. The symmetric system implies that $\mathbf{X}_2$ and $\mathbf{X}_0$ produce the same shift in the parameter space of η_1 and η_2, respectively. See [28] for an alternate derivation of different STs of a two-by-two Lax pair for P_{IV}.

5. The generators of $\widetilde{A}_2^{(1)}$

The action induced by the transformations $\mathbf{T}_1$–$\mathbf{T}_6$ is equivalent to certain combinations of generators in definition 2.1. It is then quite natural to ask whether $\mathbf{T}_1$–$\mathbf{T}_6$ are themselves composed of transformation matrices corresponding to each of the generators?

Theorem 5.1. *Consider the Lax pair (1.6) with* $\mathbf{L}$ *and* $\mathbf{M}$ *given by (1.8). Then, the action of the generators of the extended affine Weyl group* $\widetilde{A}_2^{(1)} = \langle \pi, s_0, s_1, s_2\rangle$*, is realized by the following transformations on* ψ_j*:*

$$\pi(\psi_j) = \begin{cases} z^{-1/3}\psi_{j+1}, & j = 1, 2,\\ z^{2/3}\psi_1, & j = 3, \end{cases} \tag{5.1a}$$

$$s_0(\psi_j) = \begin{cases} \psi_1 + \dfrac{\alpha_0}{z f_0}\psi_3, & j = 1, \\ \psi_j, & j \neq 1, \end{cases} \tag{5.1b}$$

$$s_1(\psi_j) = \begin{cases} \psi_2 + \dfrac{\alpha_1}{f_1}\psi_1, & j = 2, \\ \psi_j, & j \neq 2, \end{cases} \tag{5.1c}$$

$$s_2(\psi_j) = \begin{cases} \psi_3 + \dfrac{\alpha_2}{f_2}\psi_2, & j = 3, \\ \psi_j, & j \neq 3. \end{cases} \tag{5.1d}$$

Proof. The matrix corresponding to the generator π is derived in section 3, here we provide the details concerning the matrices associated with s_j. Definition 2.1 dictates that the generators of $\widetilde{A}_2^{(1)}$ satisfy the fundamental relation

$$s_j\pi = \pi s_{j+1}; \tag{5.2}$$

note that this multiplication is in the group. We have derived transformations $\mathbf{T}_1, \mathbf{T}_3, \mathbf{T}_5$, which give the corresponding action of the left-hand side of (5.2) on the Lax pair, see section 4. The right-hand side of equation (5.2), together with (2.8) and the DTs derived above, are all the ingredients required to find the matrices (5.1*b*)–(5.1*d*). Letting $\mathbf{Y}_j$ represent the action of the generator s_j, then the action of the product πs_{j+1} on the Lax pair is given by (2.8) to be

$$s_{j+1}(\mathbf{T}_0)\mathbf{Y}_{j+1}.$$

However, as $\mathbf{T}_0$ is independent of f_j and α_j we have

$$s_{j+1}(\mathbf{T}_0) = \mathbf{T}_0,$$

which, although subtle, is in fact crucial to the ensuing calculations. Substituting $j = 1$ into (5.2) gives

$$s_1\pi = \pi s_2,$$

which in terms of matrix transformations acting on $\mathbf{\Psi}$ is given by

$$\mathbf{T}_1 = \mathbf{T}_0\mathbf{Y}_2,$$

solvable for $\mathbf{Y}_2$ by a simple matrix inversion. Overall we arrive at a matrix representing the action of the generator s_2

$$\mathbf{Y}_2 = \mathbf{T}_0^{-1}\mathbf{T}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{\alpha_2}{f_2} & 1 \end{pmatrix}.$$

Similarly, substituting $j = 2, 0$ into (5.2) and using $\mathbf{T}_3$ and $\mathbf{T}_5$ respectively gives

$$\mathbf{Y}_0 = \mathbf{T}_0^{-1}\mathbf{T}_3 = \begin{pmatrix} 1 & 0 & \frac{\alpha_0}{z f_0} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{Y}_1 = \mathbf{T}_0^{-1}\mathbf{T}_5 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\alpha_1}{f_1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One can check that the matrices $\mathbf{T}_0$ and $\mathbf{Y}_j$ give the correct action of the respective generator on each entry of $\mathbf{L}$ and $\mathbf{M}$ by direct computation. Substituting (5.1) together with (1.8) into (2.6) gives the action of the generators of $\widetilde{A}_2^{(1)}$ defined by (2.2). □

6. Conclusions

The DT has provided a method of deriving the gauge transformations (5.1) presented in theorem 5.1, giving the action of the generators of $\widetilde{A}_2^{(1)}$ on the level of its associated Lax pair. The Lax pair for SP_4 is the first member of a hierarchy presented in [34] corresponding to the affine Weyl groups $\widetilde{A}_N^{(1)}$, and consequently a lot of structure is shared by all of these Lax pairs. Due to this structure, theorem 5.1 can be generalized to the remaining members of the $\widetilde{A}_N^{(1)}$ hierarchy in a straightforward way; this matter has been comprehensively dealt with in [36], and will be described in a forthcoming paper. It has recently been brought to the attention of the author that the transformations derived in [36] have also been derived using a completely different approach in [30]. Multiplying these gauge transformations has the added intricacy given by (2.8), distinguishing them from the usual notion of a representation of a group. The fact that the action of the generators as in (5.1) extends to an action of the entire group $\widetilde{A}_2^{(1)}$ can be explained in terms of non-Abelian 1-cocycle [5, 24]. Symbolically, however, the answer is neatly encompassed by the intricate multiplication rule (2.8), which describes the multiplication of any two elements—thus one is able to verify a finite number of examples. In the process, we have obtained an alternate method of constructing new STs of the SP_4 system by multiplying various DTs together.

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